Journal of Engineering Mathematics **41:** 1–11, 2001.

A note on impulsive sphere motion beneath a free-surface

T. MILOH
Faculty of Engineering, Tel-Aviv University, Ramat-Aviv, Tel-Aviv, Israel 69978

Received 12 July 2000; accepted in revised form 22 March 2001

Abstract. A general framework is presented for solving the impulsive oblique motion of a spherical body in close proximity and below a free-surface. The fluid is considered to be impulsive and the flow as incompressible. The irrotational flow field is deduced from a velocity potential. The full nonlinear problem is reduced to a sequence of boundary-value problems by employing a small-time expansion technique. The mixed boundary conditions are of a Dirichlet type on the undisturbed free-surface and of a Neumann type on the equilibrium spherical shape. The solution is obtained by employing a Green's function and the method of multipoles expansions. General expressions, correct to each order in the small-time, are given for the free-surface deflections and the pressure force experienced by the moving sphere.

Key words: impulsive flows, free-surface, hydrodynamics, multipole expansions, small-type expansions, spherical shapes.

1. Introduction

The classical hydrodynamic problem of an impulsive motion of a rigid or deformable submerged body near a free-surface is essentially nonlinear. The nonlinearity is introduced through the free-surface and impermeable body boundary conditions, whereas the field equation is taken as linear (Laplace's equation) by assuming the fluid to be inviscid and the flow incompressible. Unless the body is moving with a constant velocity in a direction parallel to the undisturbed free-surface, the problem is generally time-dependent. In many cases, such unsteady and nonlinear problems can be solved analytically by first reducing them to a sequence of linear problems by applying the method of small-time expansion (*e.g.*, Tyvand and Miloh [1]). The resulting mixed boundary value problem is of an elliptic nature and involves a general Neumann-type boundary condition applied on the deformable (instantaneous) body surface and a Dirichlet-type boundary condition on the undisturbed free-surface. Closure is supplied by enforcing a proper decay condition at infinity. The full solution of the flow problem then involves the determination of the induced velocity-potentials, free-surface deflections and hydrodynamic pressure forces evaluated to each order (in the small-time asymptotics). Solving the boundary-value problem, to order of say m, requires the successive solutions of all the boundary value problems of a lower order, *i.e.*, $0, 1, 2, \cdots, m-1$. The general procedure is demonstrated in this paper for a spherical shape and can be thus considered as a first 3-D attempt to extend similar 2-D problems of water-exit of cylindrical shapes approaching a free-surface (Tyvand and Miloh [1], Moyo and Greenhow [2]). The time-dependence of the sphere velocity, can be arbitrary and the velocity can be aligned in any direction (oblique motion). The spherical shape is also allowed to undergo small surface deformations.

The solution of the time-dependent problem is carried out by formulating a corresponding Green's function and by using the method of multipoles expansion. Similar procedures have

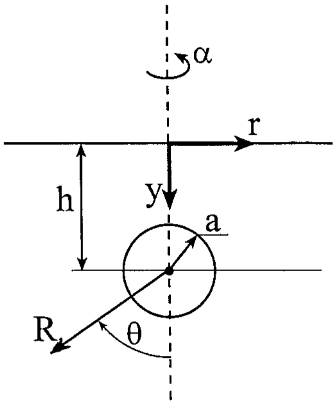

Figure 1. Definition of coordinate systems.

been employed by Srokosz [3] in the semi-bounded case and more recently by Wu [4] and Ursell [5] in analyzing the linearized wave radiation/diffraction quasi-steady problem of a submerged sphere in a channel (formulated in the frequency domain). A multipole expansion for the potential of a point source expressed in terms of Legendre polynomials is well known. The same procedure can in principle be also used for non-spherical quadratic shapes, such as spheroids or ellipsoids as long as the image singularity system of an arbitrary potential flow field within the body is known. The most general form for which a separable solution of the Laplace equation is given and such an interior image system can be analytically obtained, is the 3-D ellipsoid (Miloh [6]). However, as shown in the sequel, it is much simpler to demonstrate the proposed general methodology for a spherical shape in arbitrary oblique motions which still renders a non-symmetric 3-D problem.

2. Formulation of the problem

For this purpose let us consider a rigid or deformable sphere of equilibrium radius a whose center lies instantaneously at depth h below a free-surface (Figure 1). In formulating the boundary-value problem, it is convenient to define two sets of coordinate systems; a spherical system (R, θ, α) with an origin at the sphere's center and a cylindrical system (y, r, α) with an origin on the undisturbed free-surface $(y = 0)$, such that the center of the sphere is located at $(h, 0, 0)$. One can also use the submergence depth h as a reference length scale, thus the dimensionless sphere radius is $R = a/h = \epsilon < 1$ and its center is at $(1, 0, 0)$.

According to the small-time expansion method (Tyvand and Miloh [1]), the velocity potential ϕ, the free-surface elevation η and the hydrodynamic force F exerted on the body, can all be expressed in an asymptotic series using the time t as a small parameter, *i.e.*:

$$\begin{aligned}(\phi, \eta, F) = {}& \delta(t)(0, 0, F^{(-1)}) + H(t)\big[(\phi^{(0)}, \eta^{(0)}, F^{(0)}) + t(\phi^{(1)}, \eta^{(1)}, F^{(1)}) \\ &+ (t^2/2)(\phi^{(2)}, \eta^{(2)}, F^{(2)}) + \ldots (t^l/l!)(\phi^{(l)}, \eta^{(l)}, F^{(l)}) + \ldots\big], l = 1, 2, \ldots,\end{aligned} \tag{1}$$

where $H(t)$ and $\delta(t)$ denote the Heaviside and delta generalized functions respectively. A uniform convergence of (1) is assumed for $t \to 0^+$.

The general boundary-value problem for the velocity potential of order l, $\phi^{(l)}$ is then given by

$$\nabla^2\phi^{(l)} = 0; \quad y \geq 0 , \tag{2}$$

$$\phi^{(l)}(0, r, \alpha) = g^{(l)}(r, \alpha); \quad y = 0 , \tag{3}$$

$$\frac{\partial\phi^{(l)}}{\partial R}(R, \theta, \alpha) = f^{(l)}(\theta, \alpha); \quad R = \varepsilon , \tag{4}$$

$$| \nabla\phi^{(l)} | \to 0; \quad R \to \infty , \tag{5}$$

where the functions $f^{(l)}(\theta, \alpha)$ and $g^{(l)}(r, \alpha)$ are prescribed in terms of the velocity potentials of order less than l. The specific expressions for these boundary conditions depend on the type of impulsive motion (*e.g.,* constant velocity, constant acceleration, etc.) as well as on the deformation pattern of the spherical shape, and can be considered as given functions (see discussion in Tyvand and Miloh [1]).

In the sequel we present a general method for calculating $\phi^{(l)}$ in terms of the prescribed values of $f^{(l)}$ and $g^{(l)}$ to any order $l = 0, 1, 2, \ldots$. Also given is a scheme for evaluating the free-surface deflections $\eta^{(l)}$ and pressure forces $F^{(l)}$. We consider the general case of an oblique motion of a sphere towards or away from a free-surface. The interesting case of a vertical motion, which preserves axial-symmetry with respect to y, is readily obtained as a limiting case.

3. Solution of the boundary-value problem

In order to solve the general boundary value problem, posed in (2–5), we first define an auxiliary potential function $\bar{\phi}_m^{(l)}$ by,

$$\bar{\phi}_m^{(l)}(y, r) = \int_0^\infty k\tilde{g}_m^{(l)}(k)\, \mathrm{e}^{-ky} J_m(kr)\, \mathrm{d}k, \tag{6}$$

where J_m is the Bessel function of order m,

$$\bar{\phi}^{(l)}(y, r, \alpha) = \sum_m \bar{\phi}_m^{(l)}(y, r) \cos(m\alpha) , \tag{7}$$

$$g^{(l)}(r, \alpha) = \sum_m g_m^{(l)}(r) \cos(m\alpha) \tag{8}$$

and

$$\tilde{g}_m^{(l)}(k) = \int_0^\infty r g_m^{(l)}(r) J_m(kr)\, \mathrm{d}r . \tag{9}$$

Without loss of generality, we choose to consider only even terms in α. Here, $g_m^{(l)}$ and $\bar{\phi}_m^{(l)}$ are the Fourier coefficients of $g^{(l)}$ and $\bar{\phi}^{(l)}$ respectively, and $\tilde{g}_m^{(l)}$ is the Hankel transform of $g_m^{(l)}$. Substitution of (6) in (7) and application of (8–9) and the inverse Fourier-Bessel transform, implies that $\bar{\phi}^{(l)}(y, r, \alpha)$ is a potential function satisfying (2), (3) and (5). The Fourier coefficients $\phi_m^{(l)}(y, r)$ of the full velocity potential $\phi^{(l)}(y, r, \alpha)$ are then expressed in terms of the auxiliary potentials $\bar{\phi}_m^{(l)}$ by using a standard multipole expansion, as

$$\phi_m^{(l)}(y, r) = \bar{\phi}_m^{(l)}(y, r) - \sum_n C^{(l)}(m, n) G_n^{(m)}(y, r) \frac{\varepsilon^{n+2}}{n + 1} , \tag{10}$$

where it is understood that $m \leq n$ and the Green's function is defined as:

$$G_n^{(m)}(y,r) = \frac{P_n^{(m)}(\cos\theta)}{R^{n+1}} - \frac{(-1)^n}{(n-m)!}\int_0^\infty k^n \, \mathrm{e}^{-k(y+1)} J_m(kr)\, \mathrm{d}k \,. \tag{11}$$

Here, $C^{(l)}(m,n)$ are coefficients to be determined and $P_n^{(m)}(\cos\theta$ are the Legendre polynomials of order n and degree m. The Green's function in (11) is harmonic in the lower half space $y \geq 0$; it is singular at the origin $R = 0$ and decays for $R \to \infty$. Moreover, it can also be shown (see Hobson [7, p. 176]) that it vanishes on the undisturbed free-surface, *i.e.*, $G_n^{(m)}(0,r) = 0$.

In order to determine the unknown coefficients $C^{(l)}(m,n)$ in (10), the Neumann boundary condition (4) on the equilibrium spherical shape must be enforced. To do so, we first employ the following relationship between the cylindrical and spherical coordinate systems (Figure 1), *i.e.*: $r = R\sin\theta,\ y = 1 + R\ \cos\ \theta$, which can be verified by successive differentiations of the corresponding expansion for $m = 0$:

$$\mathrm{e}^{-ky} J_m(kr) = \mathrm{e}^{-k}\sum_{n=m}^{\infty}\frac{(-kR)^n}{(n+m)!}P_n^{(m)}(\cos\theta)\,. \tag{12}$$

Substituting then (12) in (6) and (11) we have in the immediate vicinity of the sphere

$$\bar{\phi}_m^{(l)}(R,\theta) = \sum_{N=m}\frac{P_N^{(m)}(\cos\theta)}{(N+m)!}(-R)^N\int_0^\infty k^{N+1}\,\mathrm{e}^{-k} g_m^{(l)}(k)\,\mathrm{d}k \tag{13}$$

and

$$G_n^{(m)}(R,\theta) = \frac{P_n^{(m)}(\cos\theta)}{R^{n+1}} + \sum_{N=m}\left(-\frac{1}{2}\right)^{N+n+1}\frac{(N+n)!}{(n-m)!(N+m)!}R^N P_N^{(m)}(\cos\theta)\,. \tag{14}$$

Next, we take the partial derivatives of both $\bar{\phi}_m^{(l)}$ and $G_n^{(m)}$ with respect to R evaluated at $R = \varepsilon$ and substitute (10) in (4). Making use of the orthogonality properties of the Legendre polynomials then renders the following system of linear equations for the coefficients $C^{(l)}(m,n)$, *i.e.*:

$$C^{(l)}(m,n) - \sum_{N=m}\left(-\frac{\varepsilon}{2}\right)^{N+n+1}\frac{n(N+n)!}{(N+1)(N-m)!(n+m)!}C^{(l)}(m,N) = D^{(l)}(m,n)\,, \tag{15}$$

where

$$D^{(l)}(m,n) \equiv \tilde{f}^{(l)}(m,n) + \frac{n(-\varepsilon)^{n-1}}{(n+m)!}\int_0^\infty k^{n+1}\,\mathrm{e}^{-k}\tilde{g}_m^{(l)}(k)\,\mathrm{d}k\,. \tag{16}$$

Here, $\tilde{f}^{(l)}$ represents the coefficients of the Fourier-Legendre expansion of $f^{(l)}(\theta,\alpha)$, namely

$$f^{(l)}(\theta,\alpha) = \sum_n\sum_m \tilde{f}^{(l)}(m,n)P_n^{(m)}(\cos\theta)\cos(m\alpha)\,. \tag{17}$$

Thus, once the Fourier coefficients of the boundary conditions $g^{(l)}$ (3) and $f^{(l)}$ (4) are known, the coefficients $C^{(l)}$ of the multipole expansion (10) are readily obtained by inverting the infinite set (15).

In order to show that the infinite system (15) yields a unique solution for the coefficients $C(m,n)$, it is first shown following Ursell [5, p. 613] and by introducing an index shift $N = M + m$, that the double series

$$\sum_M \sum_n (\frac{\epsilon}{3})^{M+m+n+1} \frac{n(M+m+n)!}{(M+m+1)M!(n+m)!}$$

is bounded for any m. In order to prove that it is enough to consider

$$\sum_M \sum_n (\frac{\epsilon}{2})^{M+m+n+1} \frac{(M+m+n)!}{(M+1)!(n+m-1)!} \ , \tag{18}$$

since

$$\frac{M+1}{M+m+1} \cdot \frac{n}{n+m} < 1 \quad \text{for any } (m,n)$$

Substituting again the following index shift $K = M + m + n$ in (18) and using the binomial theorem, we may show that (18) is bounded for $\epsilon < 1$, since

$$\sum_K \sum_M (\frac{\epsilon}{2})^{K+1} \frac{K!}{(M+1)!(K-M-1)!} = \frac{1}{2} \sum_K \epsilon^{K+1} \leq \frac{\epsilon}{2(1-\epsilon)} \ . \tag{19}$$

In order to complete the proof formally it is necessary to show that the double series $\sum_n \sum_m \mid D(m,n) \mid$ is also bounded. Using the definition of $D(m,n)$ (16), and assuming that both the Fourier-Bessel $\tilde{g}_m(k)$ and Fourier-Legendre $\tilde{f}(m,n)$ (17) coefficients are bounded and that the corresponding series expansions absolutely converge, we may then readily show that the double summation $\sum_n \sum_m \frac{n(n+1)!}{(n+m)!} \epsilon^{n+1}$ is bounded for $\epsilon < 1$. Finally, following Ursell's [5] methodology, we can also show that the infinite series (10) is convergent in the whole flow field.

To determine the free-surface deflection $\eta^{(l)}(r,\alpha)$, it is necessary to evaluate the normal derivative of $\phi^{(l)}$ on $y = 0$. Thus, following (6), (10) and (11) one gets,

$$\begin{aligned} \frac{\partial \phi^{(l)}(y,r)}{\partial y}\bigg|_{y=0} &= -\int_0^\infty k^2 \tilde{g}_m^{(l)}(k) J_m(kr) \, \mathrm{d}k \\ &\quad -2 \sum_n (-\varepsilon\rho)^{n+2} \left(\frac{n-m+1}{n+1} \right) C^{(l)}(m,n) P_{n+1}^{(m)}(\rho) \ , \end{aligned} \tag{20}$$

where $\rho^2 \equiv (1+r^2)^{-1}$. In deriving the second expression on the right-hand side of (20), we have used the Ferrer's integral definition of the Legendre polynomial (see for example, Whittaker and Watson [8, p. 364] or Hobson [7, p. 176]), *i.e.*

$$P_n^{(m)}(\cos\theta) = \frac{1}{(n-m)!} \int_0^\infty \mathrm{e}^{-\lambda\cos\theta} J_m(\lambda \sin\theta) \lambda^n \, \mathrm{d}\lambda \ , \tag{21}$$

which is valid for positive (m,n) and $\cos\theta > 0$. It can be easily verified that the convergence of (10) also implies the convergence of (20) since $\rho \leq 1$.

4. Hydrodynamic force

The hydrodynamic pressure force experienced by the moving sphere can be found by employing the time-dependent Bernoulli equation by integrating the induced pressure over the surface of the sphere. Thus, the lowest-order force (1), is given in terms of the zeroth-order potential $\phi^{(0)}$ as

$$\mathbf{F}^{(-1)} = \int_S \phi^{(0)} \mathbf{n} \, \mathrm{d}S \,, \tag{22}$$

where the fluid density is taken as unity and $\mathbf{n}$ denotes the inward normal to S. Higher-order terms of the force can be found by including the quadratic Bernoulli terms, *e.g.*

$$\mathbf{F}^{(l)} = \int_S \phi^{(l+1)} \mathbf{n} \, \mathrm{d}S + \sum_{l_1}^{l} \mathrm{e}_{l_1} \int_S \nabla\phi^{(l_1)} \cdot \nabla\phi^{(l-l_1)} \mathbf{n} \, \mathrm{d}S \quad l = 0, 1, 2, \dots \,, \tag{23}$$

with $e_{l_1} = \frac{1}{2}$ for $l_1 = 0$ or $l_1 = l$ and $e_{l_1} = 1$ otherwise.

In order to compute the surface integrals in (23), it is useful to expand the velocity potential $\phi^{(l)}$ in the neighborhood of the sphere in the following series of spherical harmonics;

$$\phi^{(l)}(R, \theta, \alpha) = -\sum_n \sum_m^n [A^{(l)}(m, n) R^{-(n+1)} + B^{(l)}(m, n) R^n] P_n^{(m)}(\cos\theta) \cos(m\alpha) \,. \tag{24}$$

The coefficients $A^{(l)}$ and $B^{(l)}$ are related to the coefficients $C^{(l)}$ in (10) by

$$A^{(l)}(m, n) = \frac{\epsilon^{n+2}}{n+1} C^{(l)}(m, n) \tag{25}$$

and

$$\begin{aligned} B^{(l)}(m, n) = & \sum_{N=m} (-\frac{1}{2})^{N+n+1} \frac{(N+n)!}{(n-m)!(N+m)!} \frac{\epsilon^{N+2}}{N+1} C^{(l)}(m, N) \\ & + \frac{1}{(n+m)!} \int_0^\infty (-k)^{n+1} e^{-k} g_m^{(l)}(k) \, \mathrm{d}k \,. \end{aligned} \tag{26}$$

In deriving (25–26) use has been made of (13–14) together with (10).

Of particular interest for water exit/entry problems is the evaluation of the vertical force representing the attraction/repulsion force component in a direction normal to the free-surface. The latter can be expressed in a Lagally form as a sum of three integrals:

$$\begin{aligned} F^{(l)} = & \sum_{i=1}^{3} F_i^{(l)} = \int_S \phi^{(l+1)}(\epsilon, \theta, \alpha) \cos\theta \, \mathrm{d}\theta + \sum_{l_1}^{l} e_{l_1} \int_S \frac{\partial \phi^{(l_1)}}{\partial y} f^{(l-l_1)}(\theta, \alpha) \, \mathrm{d}S \\ & - \sum_{l_1}^{l} e_{l_1} \int_\forall \frac{\partial \phi^{(l_1)}}{\partial y} \nabla^2 \phi^{(l-l_1)} \, \mathrm{d}\forall \,. \end{aligned} \tag{27}$$

The first term $F_1^{(l)}$ represents the contribution of the unsteady part. The second surface integral $F_2^{(l)}$, which involves the normal derivative of $\phi^{(l)}$ on S and the third term $F_3^{(l)}$ represented by the interior volume integral bounded by S, stem from the steady quadratic terms. Note that the existence of interior multipoles within the sphere implies that the Laplacian is not null at the location of these image singularities.

The first term $F_1^{(l)}$ in (27) can be readily obtained by using (24) and employing the orthogonality properties of the Legendre polynomials, as

$$F_1^{(l-1)} = -\frac{4\pi}{3} [A^{(l)}(0, 1) + \epsilon^3 B^{(l)}(0, 1)] \,, \tag{28}$$

which is valid for $l = 0, 1, 2, \ldots$ and thus yielding also the normal component of $\mathbf{F}^{(-1)}$ (22) for $l = 0$.

To evaluate the second integral on th right-hand-side of (27), which involves the normal derivative of the velocity potential on S, we use the following two identities for the interior and exterior spherical harmonics (see Hobson [7, p. 134]):

$$\frac{\partial}{\partial y}\left[R^n P_n^{(m)}(\cos\theta)\cos(m\alpha)\right] = (n+m)R^{n-1}P_{n-1}^{(m)}(\cos\theta)\cos(m\alpha)\ , \tag{29}$$

$$\frac{\partial}{\partial y}\left[R^{-(n+1)} P_n^{(m)}(\cos\theta)\cos(m\alpha)\right] = -(n-m+1)R^{-(n+2)}P_{n+1}^{(m)}(\cos\theta)\cos(m\alpha)\ . \tag{30}$$

Substitution of the above relationships in (27) leads to

$$\begin{aligned} F_2^{(l)} &= \sum_{l_1}^{l} e_{l_1}\int_S \frac{\partial\phi^{(l_1)}}{\partial y} f^{(l-l_1)}(\theta,\alpha)\,\mathrm{d}S \\ &= 2\pi\sum_{l_1}^{l}\sum_{n}^{\infty}\sum_{m}^{n} e_{l_1}\frac{(n+m)!}{(n-m)!}\left[\frac{n+m+1}{2n+3}\epsilon^{-n}A^{(l_1)}(m,n)\tilde{f}^{(l-l_1)}\ (m,n+1)\right. \\ &\left. - \frac{n-m}{2n-1}\epsilon^{n+1}B^{(l_1)}(m,n)\tilde{f}^{(l-l_1)}(m,n-1)\right]\ , \end{aligned} \tag{31}$$

where $\tilde{f}^{(l_1)}(m,n)$ are the Fourier-Legendre coefficient of $f^{(l_1)}(\theta,\alpha)$ defined in (17).

It should be noted that for the common Neumann problem of an impermeable surface, the normal derivative on S vanishes (*i.e.* $f^{(l)} = 0$) and thus (31) does not contribute to the traditional form of the Lagally force (*e.g.*, Landweber and Miloh [9]). However, for non-rigid (deformable) surfaces or as a consequence of the geometric nonlinearity which arises in the Lagrangian flow description of such motions (see, for example, Tyvand and Miloh [1]), this term can be rather significant and should be definitely taken into account.

Finally, we obtain an analytic expression for the third term in (27) which represents the classical ‘steady’ Lagally force resulting from the quadratic Bernoulli term. It can be expressed in terms of the various coefficients of the interior distribution of multipoles generating the outer flow (see Landweber and Miloh [9]). For the sake of completeness, we provide below a much simpler derivation of this term by taking advantage of some available theorems for spherical harmonics. Firstly, we use the well-known expansion for an exterior harmonic (using the terminology in Equation (2.4) of Miloh [10]) *i.e.*,

$$R^{-(n+1)}P_n^{(m)}(\cos\theta)\cos(m\alpha) = \frac{(-1)^n}{2(n-m)!}\frac{\partial^{n-m}}{\partial y^{n-m}}[(\frac{\partial}{\partial x}+\mathrm{i}\frac{\partial}{\partial z})^m + \text{c.c.}]\frac{1}{R}\ , \tag{32}$$

where $\theta = \cos^{-1}(y/R), \alpha = \tan^{-1}(x/z), R^2 = x^2 + y^2 + z^2$ and c.c. denotes complex conjugates. Thus, since $\nabla^2(-1/R) = 4\pi\delta(x,y,z)$ where δ is a delta function, we get from (24)

$$\nabla^2\phi^{(l_1)} = 2\pi\sum_{n}\sum_{m}^{n}A^{(l_1)}(m,n)\frac{(-1)^n}{(n-m)!}\left[\left(\frac{\partial}{\partial x}+\mathrm{i}\frac{\partial}{\partial z}\right)^m + \text{c.c.}\right]\quad\delta(x,y,z)\ . \tag{33}$$

Since all multipoles are located at the origin, the theory of generalized functions when applied to (24) and (33) implies that

$$\int_{\forall} \frac{\partial \phi^{(l_1)}}{\partial y} \nabla^2 \phi^{(l_2)} \, \mathrm{d}\forall = -\lim_{R\to 0} 2\pi \sum_n \sum_m \sum_N \sum_M A^{(l_2)}(m,n) B^{(l_1)}(M,N)$$

$$\frac{(-1)^n}{(n-m)!} \frac{\partial^{n+1-m}}{\partial y^{n+1-m}} \left[\left(\frac{\partial}{\partial x} + \mathrm{i} \frac{\partial}{\partial z} \right)^m + \text{c.c.} \right] \left\{ R^N P_N^{(M)}(\cos\theta) \cos(M\alpha) \right\} . \tag{34}$$

Lastly, we invoke Corollary 2 in Miloh [10] to (34) which results in

$$\begin{aligned} F_3^{(l)} &= -\sum_{l_1}^{l} e_{l_1} \int_{\forall} \frac{\partial \phi^{(l_1)}}{\partial y} \nabla^2 \phi^{(l-l_1)} \, \mathrm{d}\forall \\ &= 2\pi \sum_{l_1}^{l} \sum_{n}^{\infty} \sum_{m}^{n} (e_{l_1}/e_m) \frac{(n+m+1)!}{(n-m)!} A^{(l-l_1)}(m,n) B^{(l_1)}(m,n+1) , \end{aligned} \tag{35}$$

where, similar to $e_{l_1}, e_m = \frac{1}{2}$ for $m = 0$ or $m = n$ and $e_m = 1$ otherwise. This completes the derivation of the expression for the pressure force acting on the moving sphere in terms of the coefficients of the multipole expansions.

5. Example: spherical explosion near a free-surface

In order to demonstrate the preceeding analysis it may be instructive to apply it for a somewhat simplified practical case. Thus, let us consider the case of a spherical explosion near an otherwise quiescent free-surface, The instantaneous explosion at $t = 0^+$ with h denoting the epicenter depth below the free-surface is modeled by imparting an impulsive unit radial velocity to the fluid at a radial distance $R = a$. Under these conditions we wish to find $\eta^{(0)}$, the impulsively induced free-surface deflection (of a Heavyside type) and $F^{(-1)}$, the instantaneous hydrodynamic force exerted on the fictitious spherical shell $R = a$ at $t = 0^+$ (for definitions see Equation (1)). Using the notations of the general boundary value problem defined in Equations. (2–5), the present case simply corresponds to $l = 0$ (*i.e.*, zeroth order), $g^{(0)}(r,\alpha) = 0$ and $f^{(0)}(\theta,\alpha) = 1$. Since the problem is axisymmetric there is no dependence on the azimuthal angle α and thus $m = 0$.

The zeroth-order free-surface elevation is then readily given by Equation (20),

$$\eta^{(0)}(r) = -\frac{2\epsilon^2}{1+r^2} \sum_{n=0}^{\infty} \left(\frac{-\epsilon}{\sqrt{1+r^2}} \right)^n P_{n+1}\left(\frac{1}{\sqrt{1+r^2}} \right) C(n) , \tag{36}$$

where the coefficients $C(n) \equiv C^{(0)}(0,n)$, are determined from the infinite set (15), under the present simplifications, as

$$C(n) - \sum_{N=0}^{\infty} \left(-\frac{\epsilon}{2} \right)^{N+n+1} \frac{n(N+n)!}{n!(N+1)!} C(N) = \delta(N) , \tag{37}$$

where $\delta(N)$ is the Kronecker delta function.

Once the coefficients $C(n)$ are found, the impulsive force can be easily expressed in terms of these coefficients, by substituting (25) and (26) in (28), leading by virtue of (37) to

$$F^{(-1)} = -\frac{2\pi\epsilon^3}{3} \left[C(1) + \frac{\epsilon^2}{2} \sum_{N=0}^{\infty} \left(-\frac{\epsilon}{2} \right)^N C(N) \right] = -2\pi\epsilon^3 C(1) . \tag{38}$$

Table 1.

ϵ	$4C_1/\epsilon^2$	$-\eta^{(0)}(0)/2\epsilon^2$
0·1000000	0·9998743	0·9997475
0·2000000	0·9989808	0·9979196
0·3000000	0·9964805	0·9926262
0·4000000	0·9913974	0·9812745
0·5000000	0·9825560	0·9600129
0·6000000	0·9685442	0·9229063
0·7000000	0·9477606	0·8609404
0·8000000	0·9186906	0·7624471
0·9000000	0·8806725	0·6251863
0·9500000	0·8586341	0·5560364
0·9800000	0·8446388	0·5205051
0·9900000	0·8398661	0·5091420

It is interesting to note that, since $\epsilon < 1$, the infinite system can be inverted so as to yield an asymptotic recurrence type solution to any order for $C(n)$, thus $C(0) = 1$ and for $n \geq 1$, one gets

$$C(n) = n\left(-\frac{\epsilon}{2}\right)^{n+1}\left[1+\left(-\frac{\epsilon}{2}\right)^{3}\sum_{n_1=0}^{\infty}\left(-\frac{\epsilon}{2}\right)^{2n_1}\frac{n_1+1}{n_1+2}\binom{n_1+n+1}{n}[1+\right.$$
$$\left.\left(-\frac{\epsilon}{2}\right)^{3}\sum_{n_2=0}^{\infty}\left(-\frac{\epsilon}{2}\right)^{2n_2}\frac{n_2+1}{n_2+2}\binom{n_2+n_1+1}{n_1}[1+\ldots\right] \tag{39}$$

where $\binom{n}{m} \equiv \frac{n!}{m!(n-m)!}$, is the binomial coefficient.

It follows then that the impulsive free-surface deflection (step-function) to leading-order is determined from (36) as

$$\eta^{(0)}(r) = -\frac{2\epsilon^2}{(1+r^2)^{3/2}} + 0(\epsilon^3) \tag{40}$$

and the impulsive force (delta-function) is found to leading-order from (38) and (39) as

$$\bar{F}^{(-1)} = -\frac{\pi\epsilon^5}{2} + 0(\epsilon^8) \tag{41}$$

The infinite set (37) is solved first for the coefficients $C(n)$ for different values of ϵ. For $\epsilon = 0{\cdot}99$ the series is truncated after 40 terms ($N = 40$) to guarantee relative error of less than 10^{-6}. The coefficients exhibit a fast decay with N with an alternating sign. In Table 1 we present the dependence of $4C(1)/\epsilon^2$, *i.e.* the ratio of $C(1)$ to its leading-order asymptotic value $\frac{\epsilon^2}{4}$ (39) with respect to ϵ. Also given in the same Table are the values of $\eta^{(0)}(0)/2\epsilon^2$ calculated from (36). The impulsive force $F^{(-1)}$ is then determined from (38) in terms of $C(1)$. It is seen that for $\epsilon < 0{\cdot}6$, one can use the zeroth-order solutions (40) and (41) with an error of less than 10%. Plots of free-surface elevations (36) are depicted in Figure 2 as a function of the radial distance r along the free-surface for different values of ϵ. Clearly, the

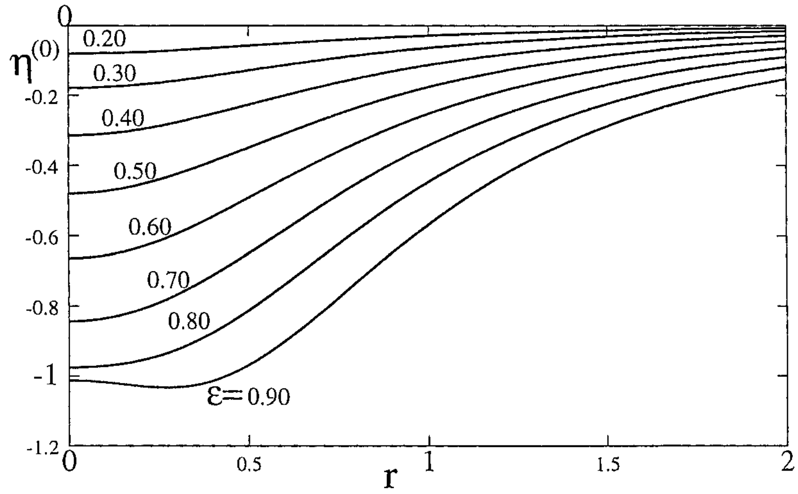

Figure 2. The variation of the zeroth-order elevation (36) with the radial distance as a function of ε.

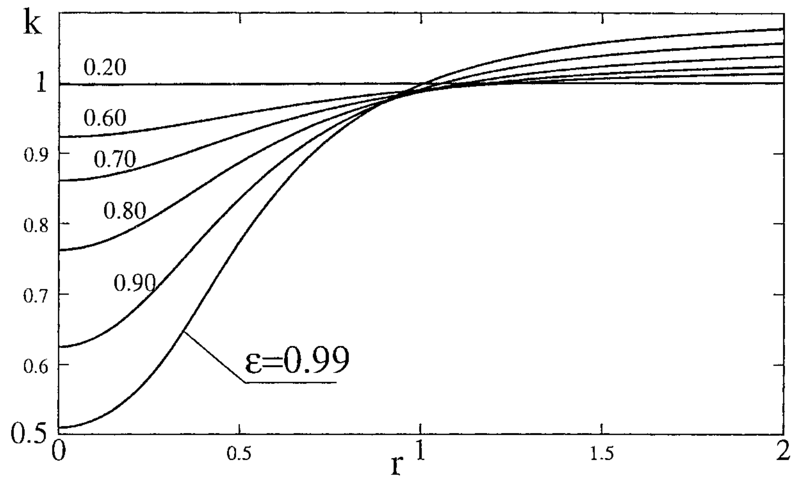

Figure 3. The ratio $k(r)$ between the leading order elevation (36) and its asymptotic value (40).

larger ϵ is, the larger is the peak/dip of the free-surface. It is also verified that for $\epsilon < 0{\cdot}6$ the free-surface can be computed by its asymptotic expression (40) with an error of less than 5 percent. For this purpose we present Figure 3 depicting the ratio $k(r)$ between $\eta^{(0)}(r)$ (36) and (40) as a function of r for $0{\cdot}2 < \epsilon < 0{\cdot}99$.

6. Concluding remarks

The general nonlinear and time-dependent hydrodynamic problem involving a deformable or rigid sphere moving impulsively near a free-surface is reduced to a sequence of linear mixed non-homogeneous boundary-value problems by using the method of small-time expansion. The general boundary-value problem for the velocity potential $\phi^{(l)}$ of order l, is given by (2–5). Here the non-homogeneous terms $f^{(l)}(\theta, \alpha)$ and $g^{(l)}(r, \alpha)$ are generally prescribed depending on the particular sphere motion and its surface deformation. A multipole expansion (10) is assumed for $\phi^{(l)}$ and the coefficients $C^{(l)}(m, n)$ are found by inverting the linear set of equations (15) depending on $f^{(l)}$ and $g^{(l)}$ through the non-homogeneous terms. It is also proven that this infinite series has a unique solution and that the corresponding expansion for the velocity potential is absolutely convergent. The free-surface deflection can then be directly calculated to arbitrary order from (20). The pressure force acting on the moving deformable sphere is found by applying the Bernoulli sum (23). The expression for the vertical force, for example, consists of three terms (27). The first is the so-called unsteady Lagally term given in (28). The other two terms result from the quadratic Bernoulli term. One of them, (21), does not appear in the traditional derivation of the Lagally theorem since in the present case $f^{(l)} \neq 0$. The last term (35) is the classical steady Lagally term for which we present here a simple and direct proof by taking advantage of some special properties of spherical harmonics.

Acknowledgement

I wish to thank F. Ursell and G. Zilman for useful discussions.

References

1. P.A. Tyvand and T. Miloh. Free-surface flow due to impulsive motion of a submerged circular cylinder. *J. Fluid Mech.* 286 (1995) 67–101.
2. S. Moyo and M. Greenhow. Free motion of a cylinder moving below and through a free-surface. *Appl. Ocean Res.* 22 (2000) 31–34.
3. M.A. Srokozz, The submerged sphere as an absorber of wave power. *J. Fluid Mech.* 95 (1979) 717–741.
4. G.X. Wu., Wave radiation and diffraction by a submerged sphere in a channel. *Quart. J. Mech. Appl. Math.* 51 (1998) 647–666.
5. F. Ursell. On the wave motion near a submerged sphere between parallel walls: Multipole potentials. *Quart. J. Mech. App. Math.* 52 (1999) 585–604.
6. T. Miloh. The ultimate image singularity system for external ellipsoidal harmonics. *SIAM. J. App. Math.* 17 (1974) 50–58.
7. E. Hobson. *The Theory of Spherical and Ellipsoidal Harmonics*. New York: Chelsea (1965).
8. E.T. Whittaker and G.N. Watson. *A Course of Modern Analysis*. Cambridge: CUP (1958).
9. L. Landweber and T. Miloh. Unsteady Lagally theorem for multipoles and deformable bodies. *J. Fluid Mech.* 96 (1980) 33–46.
10. T. Miloh. Pressure forces on deformable bodies in non-uniform impulsive flows. *Quart. J. Mech. App. Math.* 47 (1994) 635–661.